EXACT SOLUTION OF A NONSTATIONARY CONVECTIVE HEAT-EXCHANGE PROBLEM IN A TWO-DIMENSIONAL CHANNEL

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Degenerate hypergeometric functions are employed to give an exact solution of a nonstationary convective heat-conduction problem for an established laminar flow of a viscous incompressible fluid in a plane-parallel layer.

We assume that the fluid occupies the region $|x| < b$, $|y|$, $|z| < \infty$ and that the given velocity field has only the one component $v_z = -v_0\ (1 - x^2/b^2)$ (see [1]). The problem in question then amounts to determining the temperature T(x, y, z) from the equation of convective heat conduction [2]

$$\frac{\partial T}{\partial t} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right) + v_0\left(1 - \frac{x^2}{b^2}\right)\frac{\partial T}{\partial z} \tag{1}$$

under given boundary conditions and the initial condition

$$T(x,\ z,\ 0) = f(x,\ z). \tag{2}$$

If we seek particular solutions of Eq. (1) in the form

$$\exp\{[i\lambda v_0 - a(\lambda^2 + \mu^2)]t + i\lambda z\}\,\Phi(x),$$

we then obtain the following equation for the function Φ

$$\Phi'' + \left(\mu^2 - \frac{i\lambda v_0}{ab^2}x^2\right)\Phi = 0. \tag{3}$$

Assuming that boundary conditions of the first kind are homogeneous, we have, for the function Φ,

$$\Phi(\pm b) = 0. \tag{4}$$

For the solution of the boundary-value problem (3)-(4) with parameter μ we make the following substitutions:

$$\sqrt{\nu}\,\mu^2 x^2 = u,\ \Phi = \exp\left(-\frac{u}{2}\right)w(u),\ \nu = \frac{i\lambda v_0}{ab^2\mu^4}, \tag{5}$$

leading to the equation

$$uw'' + \left(\frac{1}{2} - u\right)w' - \left(\frac{1}{4} - \frac{1}{4\sqrt{\nu}}\right)w = 0, \tag{6}$$

whose general solution may be expressed in terms of degenerate hypergeometric functions [3]:

$$w = AF\left(\alpha,\ \frac{1}{2},\ u\right) + B\sqrt{u}\,F\left(\alpha + \frac{1}{2},\ \frac{3}{2},\ u\right),\ \alpha = \frac{1}{4} - \frac{1}{4\sqrt{\nu}};\ F(\alpha,\ \gamma,\ z) = \sum_{k=0}^{\infty}\frac{(\alpha)_k z^k}{(\gamma)_k k!}, \tag{7}$$

$$(\beta)_k = \beta(\beta+1)\ldots(\beta+k-1),\ (\beta)_0 = 1. \tag{8}$$

If the initial distribution is an even function of x, it is necessary to put B = 0,* after which we can write the characteristic functions of the boundary-value problem (4)-(5) in the form

* In the odd case, A = 0. The general case is handled by decomposing the function f into even and odd components.

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$$\Phi_m(x, \lambda) = F\left(\alpha, \frac{1}{2}, u\right) \exp\left(-\frac{u}{2}\right), \tag{9}$$

where the characteristic values $\mu_m(\lambda)$ are to be obtained from the equation

$$F\left(\alpha, \frac{1}{2}, u_b\right) = 0, \quad u_b = \sqrt{\frac{i\lambda v_0}{a}}\, b. \tag{10}$$

The existence of an infinite set of characteristic values follows from the asymptotic representation ($\mu_m \approx m\pi/b$, $m \to \infty$), while the discreteness of its spectrum is determined by the fact that F is an entire function of its first argument.

Thus, the formal solution of our problem is given by a double expansion (in a series involving the characteristic functions Φ_m and in a Fourier integral with respect to the variable z):

$$T = \int_{-\infty}^{\infty} \exp[i\lambda(z + v_0 t) - a\lambda^2 t]\, d\lambda \sum_m A_m(\lambda) \exp[-a\mu_m^2(\lambda)\, t]\, \Phi_m(x, \lambda), \tag{11}$$

where $A_m(\lambda)$ are functions to be found.

We can show in the usual way that the orthogonality property holds:

$$\int_{-b}^{b} \Phi_p(x, \lambda)\, \Phi_q(x, \lambda)\, \partial x = 0, \quad p \neq q.$$

Then, applying the initial condition (2), we obtain, finally,

$$A_m(\lambda) = \frac{1}{2\pi N_m(\lambda)} \int_{-b}^{b} \int_{-\infty}^{\infty} f(x, z)\, \Phi_m(x, \lambda) \exp(-i\lambda z)\, dx dz, \tag{12}$$

where

$$N_m = \int_{-b}^{b} \Phi_m^2\, dx = \frac{1}{2\mu_m} \left(\frac{\partial \Phi}{\partial x} \frac{\partial \Phi}{\partial \mu}\right)_{\substack{x=b \\ \mu=\mu_m}}. \tag{13}$$

One of the possible approaches to an approximate analysis of the exact solution obtained is to expand it in powers of a small parameter

$$\varepsilon = \sqrt{\lambda b \,\mathrm{Pe}\, i}\,. \tag{14}$$

In doing this, it may be assumed that in formula (11) it is sufficient to carry out the integration with respect to the variable λ over the interval $|\lambda b| \leq \tau^{-1/2}$, and that the condition $\sqrt{\tau} \gg \mathrm{Pe}$ is satisfied.

We employ this method in connection with the case of homogeneous boundary conditions of the second kind (thermally insulated channel walls), in which case Eqs. (4), (10), and (13) are replaced by the following:

$$\left.\frac{\partial \Phi}{\partial x}\right|_{|x|=b} = 0, \quad \left(2\frac{\partial F}{\partial u} - F\right)_{u=u_b} = 0, \quad N_m = -\frac{1}{2\mu_m} \left(\Phi \frac{\partial^2 \Phi}{\partial \mu \partial x}\right)_{\substack{x=b \\ \mu=\mu_m}},$$

where, as before, we assume that f(−x, z) = f(x, z).

Using the expansion

$$w = \sum_{p=0}^{\infty} (-1)^p (2\xi)^{2p} \varepsilon^p \sum_{m=0}^{\infty} \frac{(-1)^m}{[2(m+p)]!} a_{m,m+p} (\mu b \xi)^{2m}, \tag{15}$$

in which the coefficients a_{mk} are known from the equation

$$\left(\eta - \frac{1}{4}\right)\left(\eta - \frac{5}{4}\right) \cdots \left(\eta - \frac{4k-3}{4}\right) = \sum_{m=0}^{k} a_{mk} \eta^m, \quad a_{00} = 1,$$

we obtain the following approximate expressions for the characteristic values and characteristic functions:

$$b\mu_m = m\pi + \left(\frac{1}{6} + \frac{1}{4\pi^2 m^2}\right) \frac{\varepsilon^2}{m\pi} + O(\varepsilon^4), \quad m \geqslant 1, \tag{16}$$

$$(b\mu_0)^2 = \left(\frac{1}{3} - \frac{8}{945}\varepsilon^2\right)\varepsilon^2 + O(\varepsilon^6) = (b\mu_{00})^2 + O(\varepsilon^6), \tag{17}$$

$$\Phi_m = \cos m\pi\xi + \left[\left(\frac{\xi^2 - 1}{3m\pi} - \frac{1}{m^3\pi^3}\right)\xi \sin m\pi\xi + \frac{\xi^2 \cos m\pi\xi}{2m^2\pi^2}\right]\frac{\varepsilon^2}{2} + O(\varepsilon^4), \quad m \geqslant 1, \tag{18}$$

$$\Phi_0 = 1 + \xi^2(\xi^2 - 2)\frac{\varepsilon^3}{12} + O(\varepsilon^4). \tag{19}$$

The latter expressions make it possible, in particular, to obtain the following expression for the temperature averaged over a channel section:

$$\overline{T}(z, t) = \frac{1}{2b}\int_{-b}^{b} T(x, z, t)dx = \int_{-\infty}^{\infty} \exp[i\lambda(z + v_0 t) - a(\lambda^2 + \mu_{00}^2)t][G(\lambda) + O(\varepsilon^2)]\,d\lambda,$$

$$G(\lambda) = \frac{1}{4\pi b}\int_{-b}^{b}\int_{-\infty}^{\infty} f(x, z)\exp(-i\lambda z)\,dxdz, \tag{20}$$

from which it is evident that the function $\overline{T}$ satisfies to within quantities of order Pe $\tau^{-1/2}$ the differential equation

$$\frac{\partial \overline{T}}{\partial t} = (a + a_k)\frac{\partial^2 \overline{T}}{\partial z^2} + \bar{v}\frac{\partial \overline{T}}{\partial z}, \quad a_k = \frac{8}{945}\mathrm{Pe}^2 a. \tag{21}$$

It is natural to call the quantity a_k the convective thermal diffusivity coefficient by analogy with the convective diffusion coefficient introduced in [4] in connection with an approximate consideration of the corresponding diffusion processes.

We note, in conclusion, that the method we have presented for obtaining exact solutions of nonstationary problems of convective heat transfer (and diffusion) can be extended to problems with nonhomogeneous boundary conditions (including even problems of the third kind) as well as to more involved problems of junction heat exchange (see, for example, [5] in which an approximate solution is given of a junction problem for a many-layered semiinfinite channel without taking account of axial spreading of heat).

NOTATION

x, y, z, coordinates; $v_0 = qb^2/2\mu'$, maximum velocity; q, constant pressure drop per unit length; μ', viscosity coefficient; t, time; a, coefficient of thermal diffusivity; Pe = $v_0 b/a$, Peclet number; $\tau = at/b^2$, dimensionless time; $\xi = x/b$, dimensionless coordinate; $\bar{v} = 2/3\, v_0$, averaged velocity.

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